

NSF-ITP-99-085

hep-th/9907166

Non-Commutative Yang-Mills and The AdS/CFT Correspondence

Akikazu Hashimoto

Institute for Theoretical Physics
University of California, Santa Barbara, CA 93106
aki@itp.ucsb.edu

N. Itzhaki

Department of Physics
University of California, Santa Barbara, CA 93106
sunny@physics.ucsb.edu

Abstract

We study the non-commutative supersymmetric Yang-Mills theory at strong coupling using the AdS/CFT correspondence. The supergravity description and the UV/IR relation confirms the expectation that the non-commutativity affects the ultra-violet but not the infra-red of the Yang-Mills dynamics. We show that the supergravity solution dual to the non-commutative $\mathcal{N} = 4$ SYM in four dimensions has no boundary and defines a minimal scale. We also show that the relation between the B field and the scale of non-commutativity is corrected at large coupling and determine its dependence on the 't Hooft coupling λ .

July 1999

Classical supersymmetric Yang-Mills theory (SYM) in $(p + 1)$ -dimensional space can be generalized to SYM on non-commutative spaces [1]. Since the generalization involves infinitely many higher order terms, it is very hard to provide a pure field theory proof that such a theory is consistent at the quantum level. String theory provides a way to obtain these theories by considering the decoupling limit of $D(p - 2)$ -branes in type II string theories on T^2 with a background NSNS 2-form field $B_{\mu\nu}$ polarized along the plane of the torus [2, 3]. The fact that non-commutative SYM is obtained from string theory, in a limit which does not involve gravity, suggests (if string theory with constant B field is consistent) that the non-commutative SYM, at least with sixteen supercharges, is a consistent theory at the quantum level.

In this article, we take advantage of the fact that exactly the same decoupling limit leads to the near horizon geometry of the Dp -branes [4, 5] to learn about the non-commutative SYM at large coupling¹. We begin by reviewing the argument of [2, 3] regarding how the background B -field gives rise to a non-commutativity of scale Δ . We describe a scaling limit on the field theory side which keeps the Δ finite while sending α' to zero to decouple the stringy excitations. Then, we consider the same scaling limit in the dual closed-string picture, and find that the background B -field changes the dynamics of the closed strings only in the region far away from the horizon (UV), while keeping the dynamics near the horizon (IR) unaffected. We find that the background geometry does not have any boundary, which we interpret as the manifestation of the fact that theories on non-commutative spaces do not have a local UV description.

Following [3], we consider $D(p - 2)$ -branes in a weakly coupled type II string theory, oriented along the x_0, \dots, x_{p-2} directions. Consider compactifying x_{p-1} and x_p coordinates on a square torus of radius R and turn on a constant NSNS B -field polarized along the plane of this torus. In the absence of the B -field and in the limit

$$\frac{R}{\alpha'} = \frac{1}{\Sigma} = \text{fixed}, \quad \alpha' \rightarrow 0, \quad (1)$$

it is natural to describe this system in the T-dual picture of Dp -branes wrapping the dual torus whose size Σ is macroscopic.

As we shall see shortly, to obtain a finite non-commutativity scale² in the decoupling limit, the B field has to satisfy

$$\Delta^2 = B\alpha' = \text{fixed}, \quad \alpha' \rightarrow 0. \quad (2)$$

Thus the B field has to be very large, and in the presence of such a strong B -field, the T-duality is strongly modified. Note that since we would like to make contact with the

¹Related ideas were discussed in [2, 6].

²In the estimate of scales, numerical factors of order one are ignored.

AdS/CFT correspondence we are using conventions which are natural from the supergravity point of view. These conventions are different then the ones used in the recent non-commutative geometry literature. In our conventions $S_B = \frac{1}{4\pi\alpha'} \int d\sigma^2 \epsilon^{\alpha\beta} B_{\mu\nu} \partial_\alpha x^\mu \partial_\beta x^\nu$ where x has the dimension of length (if x is compactified then $x \sim x + 2\pi R$) and hence B and G are dimensionless. The fact that a large B field strongly modifies the T-duality transformation can be seen from the form of the transformation of the matrix E [7] which in our notations takes the following form:

$$E = \frac{R^2}{\alpha'}(G + B) = \frac{R^2}{\alpha'} \begin{pmatrix} 1 & B \\ -B & 1 \end{pmatrix}. \quad (3)$$

T-duality takes E to E^{-1} and so the dual radius in the presence of B field is

$$\Sigma_B = \Sigma \frac{\alpha'}{\Delta^2} \quad (4)$$

which is not macroscopic, as Σ_B vanishes when $\alpha' \rightarrow 0$. Therefore, in the presence of a large B field, we cannot T-dualize to end up with Dp -branes wrapping $p + 1$ macroscopic dimensions. Instead we end up with only $p - 1$ macroscopic dimensions.

On the other hand, due to the Dirichlet boundary conditions along the x_{p-1} and the x_p directions, the energy

$$E_s = \frac{1}{\Sigma} \quad (5)$$

of a string stretched between the images of the branes is not affected by the B field. Therefore, from the point of view of the open strings living on the $D(p-2)$ -branes, the theory has $p + 1$ macroscopic dimensions.

We have concluded that although the D-branes are wrapping only $p - 1$ large directions, the field theory living on the branes knows about $p + 1$ large dimensions! This apparent mismatch of the number of macroscopic dimensions is disturbing in light of the AdS/CFT conjecture which implies a duality between the closed string and open string description of the branes. The goal of this article is to resolve this discrepancy and to provide an interpretation of the non-locality in the dual closed string picture. To achieve this goal, it is useful to first examine the relation between the B -field and the non-commutativity scale Δ more closely [3].

Instead of T-dualizing twice, Douglas and Hull considered the following chain of “duality” transformations. First, they perform a T-duality along one of the cycles. Due to the presence of the B -field in the background, the dual torus will not be rectangular. The $D(p-2)$ -branes have now become $D(p-1)$ -branes and the light degrees of freedom of equation (5) are now momentum modes along the $D(p-1)$ -branes and the winding modes along the short cycle.

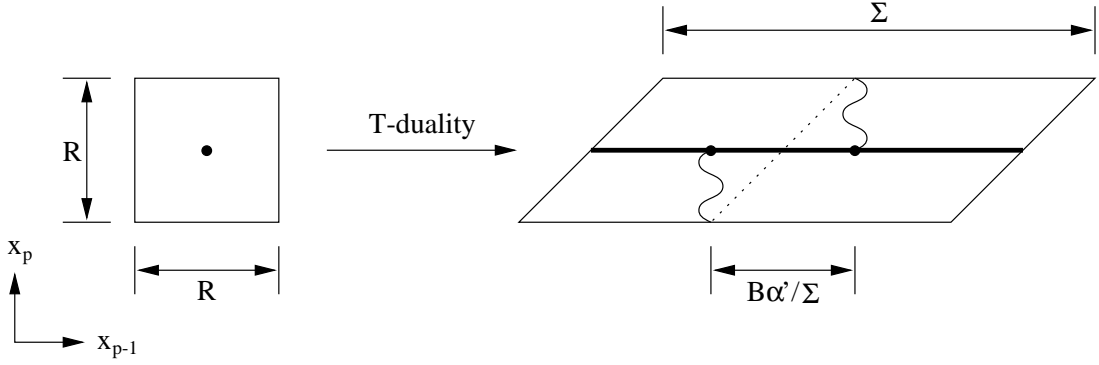


Figure 1: $D(p-2)$ -branes in square torus of radius $R \ll \sqrt{\alpha'}$ and B-field flux and its T-dual $D(p-1)$ -branes on a skewed torus. Skewed geometry of the torus gives rise to non-locality in the open string excitations living on the $D(p-1)$ -brane.

To minimize the energy, winding modes will wind the torus in the shortest path illustrated in figure 1. Therefore, both the momentum and the winding modes have masses of order $1/\Sigma$. Due to the skewed shape of the dual torus, however, the winding modes are delocalized along the $D(p-1)$ -brane world volume by length of order

$$\Delta_{p-1} = \frac{B\alpha' w}{\Sigma}, \quad (6)$$

where w is the winding number. The Compton wavelength along the x_p direction associated with such a state is of order $\Delta_p = w/\Sigma$. Combining these results, the scale of non-locality comes out to

$$\Delta^2 \equiv \Delta_{p-1} \Delta_p = B\alpha'. \quad (7)$$

Readers are referred to [3, 8] for more details.

Since the B field has a finite effect on the field theory living on the brane in the limit (1) and (2), it should also have a finite effect on the dual supergravity description of the theory. At first sight this does not seem to be the case, since on the closed string side of the duality we do not end up with $p+1$ large dimensions. The resolution stems from taking proper account of the effect of the D-brane background geometry in the near horizon region. For concreteness, let us concentrate on the conformal case by setting $p=3$. Our conclusions can be generalized immediately to the non-conformal cases with $p \neq 3$. The string frame solution in the presence of D1-branes and their images coming from the T^2 compactification is [9]³

$$ds^2 = f^{-1/2}(-dt^2 + dx_1^2) + f^{1/2}(dx_2^2 + \dots + dx_9^2),$$

³The solution is not modified by a constant B -field since $H = dB = 0$ and does not act as a source for the other supergravity fields.

$$\begin{aligned} e^\phi &= f^{1/2}, \\ B_{12} &= \frac{\Delta}{\alpha'}, \end{aligned} \tag{8}$$

where f is the harmonic function of the transverse coordinates $U = \sqrt{x_4^2 + \dots + x_9^2}/\alpha'$ and we Poisson re-sum over the images coming from the T^2 directions x_2 and x_3 ,

$$f = 1 + \frac{\lambda}{\alpha'^2 U^4}. \tag{9}$$

The 't Hooft coupling constant of the four dimensional field theory is denoted by $\lambda = 4\pi g_s N$. We see that in the near horizon region the longitudinal directions shrinks while the transverse directions blow up. Proper treatment of this effect amounts to setting g_{22} and g_{33} equal to $f^{1/2}$ instead of 1 in equation (3). Therefore, in the near horizon limit, g_{22} and g_{33} blow up just at the right strength to compete with effect of the B field. Applying T-duality to this background⁴ and taking the field theory decoupling limit, we obtain

$$\begin{aligned} ds^2 &= \alpha' \left\{ \frac{U^2}{\sqrt{\lambda}} (-dt^2 + dx_1^2) + \frac{\sqrt{\lambda} U^2}{\lambda + U^4 \Delta^4} (dx_2^2 + dx_3^2) + \frac{\sqrt{\lambda}}{U^2} dU^2 + \sqrt{\lambda} d\Omega_5^2 \right\}, \\ e^\phi &= \frac{\lambda}{4\pi N} \sqrt{\frac{\lambda}{\lambda + \Delta^4 U^4}}, \\ B_{12} &= -\frac{\alpha' \Delta^2 U^4}{\lambda + \Delta^4 U^4}, \end{aligned} \tag{10}$$

with periodicities $x_2 \sim x_2 + \Sigma$ and $x_3 \sim x_3 + \Sigma$. To avoid the finite size effects we take $\Sigma \gg \Delta$. In the spirit of [4] we conjecture that the type IIB string theory on this background is dual to non-commutative SYM with non-commutative x_2 - x_3 plane.

Equation (10) is the main result of this paper. It describes the dual supergravity background corresponding to the same scaling limit used to define SYM on non-commutative geometries. Let us pause and make a few comments on the qualitative features of (10).

- The geometry (10) is the effective description of our system when the curvature and the coupling are small. According to (10), the dilaton is small everywhere in the large N limit. Unlike in AdS the invariant curvature in string units depends on U . However, it is always of the order of the AdS curvature, $1/\sqrt{\lambda}$. Thus for large 't Hooft coupling we can trust the solution everywhere. Notice that after the T-duality the B field is not a constant and hence $H \neq 0$.
- The observation of [4] that U plays the role of energy scale on the field theory side is not modified by Δ as the energy of a string stretched between the collection of the branes and a probe brane is the same as in the AdS case. This follows from the fact that Δ does not

⁴Under T-duality, the dilaton transforms according to $\phi' = \phi - \frac{1}{4} \log(\det g / \det g')$ [7].

modify the relation $\sqrt{g_{tt}g_{UU}} = \alpha'$ which determines the energy of the string. Alternatively, in the D1-branes language, before the T-duality, the presence of the B field does not modify the energy of the open strings.

- The isometries of equation (10) are $SO(1,1) \times SO(2) \times SO(6)$. $SO(1,1) \times SO(2)$ and the translation invariance are the remnants of the $SO(4,2)$ of AdS_5 . The fact that the conformal and special conformal transformation are broken follows from the presence of the scale Δ . The fact that Lorentz invariance is broken to $SO(1,1) \times SO(2)$ agrees with the effect of equation (7) on the field theory side. The $SO(6)$ is the isometry of the 5-sphere, and corresponds to the $SU(4)$ R-symmetries of the $\mathcal{N} = 4$ supersymmetry algebra. The fact $SU(4)$ is not broken by the non-commutativity implies that the supersymmetry is not broken by the non-commutativity either. Note that since the conformal invariance is broken the number of supercharges is 16 and not 32. Furthermore, the background is not self-dual with respect to S-duality.

- The presence of a finite non-locality scale implies that the dynamics at large distances (compared to the non-locality scale) is not affected while the short distances dynamics is drastically changed. This is exactly what we see in the supergravity description. Equation (10) describes the usual $AdS_5 \times S^5$ solution with a constant dilaton in the IR ($U \rightarrow 0$), while the solution is strongly modified in the UV. On the supergravity side the non-commutativity scale can be read off from the point at which the modification to the $AdS_5 \times S^5$ background becomes of order one. This happens at

$$U = \frac{\lambda^{1/4}}{\Delta}. \quad (11)$$

Using the UV/IR relation [10, 11], $L \sim \sqrt{\lambda}/U$, we find that at large 't Hooft coupling the non-commutativity scale is not $\Delta = B\alpha'$ but rather⁵

$$\tilde{\Delta} = \Delta\lambda^{1/4} = B\alpha'\lambda^{1/4}. \quad (12)$$

The fact that the $\tilde{\Delta}$ is different than Δ is an indication that the relation between B and the non-commutativity scale receives quantum corrections. It would be interesting to study the corrections in perturbation theory.

Although the discussion above provides some evidence that the theory acquires a new dynamical scale at $\lambda^{1/4}\Delta$, we have not yet demonstrated (other than by construction) that this scale is associated with non-commutative geometry. Non-commutative geometry has a built in minimal distance scale, and we would like to see this from the supergravity point of view. A clean way to see that there is such a minimal distance is the following. Let us add

⁵The general expression for arbitrary p is $\tilde{\Delta} = \Delta^{2(5-p)/(7-p)}\lambda^{1/(7-p)}$.

a non-commutativity scale in the x_0 - x_1 plane (by starting with type IIB D-instantons in the presence of B_{23} and B_{01}). The corresponding supergravity solution (in the Euclidean space) is

$$ds^2 = \alpha' \left\{ \frac{\sqrt{\lambda}U^2}{\lambda + U^4\Delta_{01}^4}(dx_0^2 + dx_1^2) + \frac{\sqrt{\lambda}U^2}{\lambda + U^4\Delta_{23}^4}(dx_2^2 + dx_3^2) + \frac{\sqrt{\lambda}}{U^2}dU^2 + \sqrt{\lambda}d\Omega_5^2 \right\}. \quad (13)$$

To simplify the discussion, let us set $\Delta_{01} = \Delta_{23} = \Delta^6$. This geometry is manifestly invariant under the transformation

$$\tilde{U} = \frac{\lambda^{1/2}}{U\Delta^2}. \quad (14)$$

As we mentioned earlier, this geometry asymptotes to $AdS_5 \times S_5$ in the small U (IR) limit. Thus from the IR point of view the boundary should be at $U = \infty$. However, as a result of the $U \leftrightarrow \tilde{U}$ invariance, the $U \rightarrow \infty$ limit is also $AdS_5 \times S_5$. So from the UV point of view the boundary should be at $\tilde{U} = \infty$ and therefore at $U = 0$. In fact the space described by eq.(13) is essentially two AdS spaces which are glued together in such a way that geodesics starting from the region near the horizon of one of the AdS spaces reach the interior of the other AdS, and so this space has no boundary⁷. Now, a local field theory is defined at short distances, and in terms of the AdS/CFT correspondence this means that the microscopic structure of the theory is encoded on the boundary of the AdS space. Having a non-commutative theory would imply that we should not be able to define the theory at short distances. The fact that our geometry has no boundary is the supergravity manifestation of this fact, and the minimal distance scale is set by the self-dual point $U = \lambda^{1/4}\Delta$, in agreement with equation (12).

The symmetry (14) resumables T-duality of string theory, with $\Delta^2/\lambda^{1/2}$ plays the role of α' . This analogy might be helpful in future investigation of non-commutative geometry.

The goal of this investigation was to understand the mechanism of non-locality in the non-commutative SYM at large gauge coupling from the dual supergravity description. We were guided by the intuition that when the effect of non-locality of order Δ in SYM is turned on, the dynamics at length scales longer than Δ is unaffected, whereas the dynamics at length scales smaller than Δ is drastically changed. We formulated a scaling limit of open string dynamics which keeps Δ fixed while sending α' to zero. Following the same scaling limit and applying T-duality, we obtained the background geometry (10). In the $U \rightarrow 0$ limit, this geometry asymptotes to the usual $AdS_5 \times S_5$ geometry, confirming our expectation that the IR dynamics is unaffected by non-commutativity. As U is increased, the geometry starts to deviate from its AdS limit. Because of the $U \leftrightarrow \tilde{U}$ invariance, the geometry in

⁶This case is the one relevant for [12].

⁷The background (10), corresponding to the case of vanishing Δ_{01} , does have a boundary at $U = \infty$ but with only two dimensions parameterized by t and x_1 .

the $U \rightarrow \infty$ limit is also an $AdS_5 \times S_5$ geometry, whose natural radial coordinate is \tilde{U} . We find therefore that the supergravity dual to the non-commutative SYM does not have a boundary, unlike the previously encountered examples of the boundary/bulk correspondence. Although this might seem surprising at first sight, it follows quite naturally from the fact that field theories on non-commutative spaces do not admit a local UV description. The scale of non-commutativity $\tilde{\Delta} = \lambda^{1/4} \Delta$ can be read off from the self-dual scale of the $U \leftrightarrow \tilde{U}$ transformation. This non-commutativity scale disagrees with the non-commutativity scale computed in the weakly coupled limit in [3] by a certain functional dependence on the 't Hooft coupling constant λ . We interpret this to mean that the relation between Δ and the background B field receives quantum corrections.

Acknowledgments

NI would like to thank S. Yankielowicz for discussions. The work of AH is supported in part by the National Science Foundation under Grant No. PHY94-07194. The work of NI is supported in part by NSF grant No. PHY9722022.

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